Strongly Self-Absorbing C^*-algebras which contain a nontrivial projection

Marius Dadarlat* and Mikael Rørdam†

Dedicated to Joachim Cuntz on the occasion of his 60th birthday

Abstract

It is shown that a strongly self-absorbing C^*-algebra is of real rank zero and absorbs the Jiang-Su algebra if it contains a non-trivial projection. We also consider cases where the UCT is automatic for strongly self-absorbing C^*-algebras, and K-theoretical ways of characterizing when Kirchberg algebras are strongly self-absorbing.

1 Introduction

Strongly self-absorbing C^*-algebras were first systematically studied by Toms and Winter in [11]. The classfication program of Elliott had prior to that been seen to work out particulaly well for those (separable, nuclear) C^*-algebras that tensorially absorb one of the Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, or the Jiang-Su algebra $\mathcal{Z}$. More precisely, thanks to deep theorems of Kirchberg, the classification of separable, nuclear, stable C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_2$ is complete (the invariant is the primitive ideal space); and separable, nuclear, stable C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_\infty$ are classified by an ideal related KK-theory. The situation for separable, nuclear C^*-algebras that absorb the Jiang-Su algebra is at present very promising (see for example [12]) but not as complete as in the purely infinite case.

The C^*-algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$ and $\mathcal{Z}$ are all examples of strongly self-absorbing C^*-algebras. They are in [11] defined to be those unital separable C^*-algebras $D \neq \mathbb{C}$ for which there is an isomorphism $D \to D \otimes D$ that is approximately unitarily equivalent to the *-homomorphism $d \mapsto d \otimes 1$. Strongly self-absorbing C^*-algebras are automatically simple and nuclear, and they have at most one tracial state. It is shown in [11] that if D is a strongly self-absorbing C^*-algebra in the UCT class, then it has the same K-theory as one of the C^*-algebras in the following list: $\mathcal{Z}$, UHF-algebras of infinite type, $\mathcal{O}_\infty$, $\mathcal{O}_\infty$ tensor

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a UHF-algebra of infinite type, or $\mathcal{O}_2$. It is an open problem if nuclear C^*-algebras always satisfy the UCT (and also if strongly self-absorbing C^*-algebras enjoy this property); and it is an intriguing problem, very much related to the Elliott classification program, if the list above exhausts all strongly self-absorbing C^*-algebras. Should the latter be the case, then it would in particular follow that every strongly self-absorbing C^*-algebra absorbs the Jiang-Su algebra $\mathcal{Z}$. By the Kirchberg-Phillips classification theorem, a strongly self-absorbing Kirchberg algebra belongs to the list above if and only if it belongs to the UCT class. Let us also remind the reader that a strongly self-absorbing C^*-algebra is a Kirchberg algebra if and only if it is not stably finite (or, equivalently, if and only if it is traceless).

In Section 2 of this paper we show that every strongly self-absorbing C^*-algebra which contains a non-trivial projection is of real rank zero and absorbs the Jiang-Su algebra. In Section 3 we consider K-theoretical conditions on strongly self-absorbing Kirchberg algebras. One such condition (phrased at the level of K-homology) characterizes the Kirchberg algebra $\mathcal{O}_\infty$, and other results in Section 3 give K-theoretical characterizations on when a Kirchberg algebra is strongly self-absorbing.

2 Strongly self-absorbing C^*-algebras with a non-trivial projection

In this section we show that any strongly self-absorbing C^*-algebra that contains a non-trivial projection is automatically approximately divisible, of real rank zero, and absorbs the Jiang-Su algebra $\mathcal{Z}$

Lemma 2.1 *There is a unital *-homomorphism from $M_3 \oplus M_2$ into a unital C^*-algebra A if and only if A contains projections e, e' such that $e \perp e'$, $e \sim e'$, and $1 - e - e' \precsim e$.*

Proof: It is easy to see that such projections e and e' exist in $M_3 \oplus M_2$ and hence in any unital C^*-algebra A that is the target of a unital *-homomorphism from $M_3 \oplus M_2$.

Assume now that such projections e and e' exist. Let $v \in A$ be a partial isometry such that $v^*v = e$ and $vv^* = e'$. Put $f_0 = 1 - e - e'$. Find a subprojection f_1 of e which is equivalent to f_0, and put $f_2 = vf_1v^*$. Put $g_1 = e - f_1$ and put $g_2 = e' - f_2 = vg_1v^*$. The projections f_0, f_1, f_2, g_1, g_2 then satisfy

$$1 = f_0 + f_1 + f_2 + g_1 + g_2, \qquad f_0 \sim f_1 \sim f_2, \qquad g_1 \sim g_2.$$

Extending the sets $\{f_0, f_1, f_2\}$ and $\{g_1, g_2\}$ to sets of matrix units for M_3 and M_2, respectively, yields a unital *-homomorphism from $M_3 \oplus M_2$ into A. (If the f_j's are zero or if the g_j's are zero, then this *-homomorphism will fail to be injective, and will instead give a unital embedding of M_2 or M_3 into A.) □

If D is any unital C^*-algebra then we let $D^{\otimes n}$ denote the n-fold tensor product $D \otimes D \otimes \cdots \otimes D$ (with n tensor factors), and we let $D^{\otimes \infty}$ denote the infinite tensor product $\bigotimes_{n=1}^{\infty} D$. The latter is the inductive limit of the sequence

$$D \to D^{\otimes 2} \to D^{\otimes 3} \to D^{\otimes 4} \to \cdots,$$

(with connecting mappings $d \mapsto d \otimes 1_D$). We shall view D as a (unital) sub-C^*-algebra of $D^{\otimes n}$, $D^{\otimes n}$ as a sub-C^*-algebra of $D^{\otimes m}$ (if $n \leq m$), and finally D and $D^{\otimes n}$ are viewed as subalgebras of $D^{\otimes \infty}$.

If $x \in D^{\otimes n}$, then $x^{\otimes k}$ will denote the k-fold tensor product

$$x^{\otimes k} = x \otimes x \otimes x \otimes \cdots \otimes x \in D^{\otimes kn}.$$

The proof of the lemma below resembles the proof of [9, Lemma 6.4].

Lemma 2.2 *Let D be a strongly self-absorbing C^*-algebra, and let p be a projection in D. Consider the following projections in $D \otimes D$,*

$$e_1 = p \otimes (1-p), \qquad e_1' = (1-p) \otimes p, \qquad f = p \otimes p + (1-p) \otimes (1-p).$$

For each natural number n consider also the following projections in $D^{\otimes 2(n+1)}$,

$$e_{n+1} = f^{\otimes n} \otimes p \otimes (1-p), \qquad e_{n+1}' = f^{\otimes n} \otimes (1-p) \otimes p.$$

It follows that the projections $e_1, e_2, \ldots, e_1', e_2', \ldots$ are pairwise orthogonal in $D^{\otimes \infty}$, and that $e_j \sim e_j'$. Moreover, for each natural number n, set

$$E_n = e_1 + e_2 + \cdots + e_n, \qquad E_n' = e_1' + e_2' + \cdots + e_n'.$$

Then $E_n \perp E_n'$, $E_n \sim E_n'$, and

$$1 - E_n - E_n' = f^{\otimes n}. \tag{2.1}$$

Proof: The equivalence $e_j \sim e_j'$ comes from the fact that the flip automorphism $a \otimes b \mapsto b \otimes a$ on $D \otimes D$ is approximately inner when D is strongly self-absorbing. The projections $e_1, e_2, \ldots, e_1', e_2', \ldots$ are pairwise orthogonal by construction. The only thing left to prove is (2.1). We prove this by induction after n, and note first that (2.1) for $n = 1$ follows from the fact that $e_1 + e_1' + f = 1$. Suppose that (2.1) holds for some $n \geq 1$. Then

$$\begin{aligned} 1 - E_{n+1} - E_{n+1}' &= 1 - E_n - E_n' - e_{n+1} - e_{n+1}' \\ &= f^{\otimes n} \otimes 1_D \otimes 1_D - f^{\otimes n} \otimes p \otimes (1-p) - f^{\otimes n} \otimes (1-p) \otimes p \\ &= f^{\otimes (n+1)}. \end{aligned}$$

□

Lemma 2.3 *Let D be a strongly self-absorbing C^*-algebra and let p be a projection in D such that $p \neq 1$. Then there exists a natural number n such that $p^{\otimes n} \precsim 1 - p^{\otimes n}$ in $D^{\otimes n}$.*

Proof: To simplify the notation we express our calculations in terms of the monoid $V(D)$ of Murray-von Neumann equivalence classes of projections in D and in matrix algebras over D. Let $[e] \in V(D)$ denote the equivalence class containing the projection e in (a matrix algebra over) D.

Since D is simple and $p \neq 1$ there is a natural number n such that $n[1-p] \geq [p]$. It follows that

$$\begin{aligned}[1-p^{\otimes n}] &\geq [(1-p)\otimes p \otimes \cdots \otimes p] + [p \otimes (1-p) \otimes \cdots \otimes p] + [p \otimes p \otimes \cdots \otimes (1-p)] \\ &= n[(1-p)\otimes p \otimes \cdots \otimes p] \\ &\geq [p \otimes p \otimes p \otimes \cdots \otimes p] = [p^{\otimes n}],\end{aligned}$$

where the equality between the second and third expression holds because the flip on a strongly self-absorbing C^*-algebra is approximately inner. $\square$

Lemma 2.4 *Let D be a strongly self-absorbing C^*-algebra, let p be a projection in $D^{\otimes k}$, and let e be a projection in $D^{\otimes l}$ for some natural numbers k and l. Assume that $p \neq 1$ and that $e \neq 0$. It follows that there exists a natural number n such that $p^{\otimes n} \precsim e$ in $D^{\otimes \infty}$.*

Proof: Let d be a natural number such that $dk \geq l$. Upon replacing p with $p^{\otimes d}$, e with $e \otimes 1_D^{\otimes(dk-l)}$, and D with $D^{\otimes dk}$ we can assume that p and e both belong to D. Use Lemma 2.3 to find m such that $p^{\otimes m} \precsim 1 - p^{\otimes m}$. By replacing p with $p^{\otimes m}$, e with $e \otimes 1_D^{\otimes(m-1)}$, and D with $D^{\otimes m}$ we can assume that p and e both belong to D and that $p \precsim 1-p$.

Now, $p \sim q \leq 1-p$ for some projection q in D. In the language of the monoid $V(D)$ we have

$$[1_D{}^{\otimes k}] \geq [(p+q)^{\otimes k}] = 2^k[p^{\otimes k}]$$

for any natural number k. Using simplicity of D we can choose n such that $2^{n-1}[e] \geq [p]$. Then

$$[e] = [e \otimes 1_D{}^{\otimes(n-1)}] \geq 2^{n-1}[e \otimes p^{\otimes(n-1)}] \geq [p^{\otimes n}],$$

in $V(D^{\otimes n})$ as desired, where we in the first identity have used that the embedding of D into $D^{\otimes n}$ maps e onto $e \otimes 1_D{}^{\otimes(n-1)}$. $\square$

Theorem 2.5 *Let D be a strongly self-absorbing C^*-algebra. Then the following three conditions are equivalent:*

(i) *D contains a non-trivial projection (i.e., a projection other than 0 and 1).*

(ii) *D is approximately divisible.*

(iii) *D is of real rank zero.*

If any of the three equivalent conditions are satisfied, then D absorbs the Jiang-Su algebra, i.e., $D \cong D \otimes \mathcal{Z}$.

Proof: (i) $\Rightarrow$ (ii). If D is strongly self-absorbing, then there is an asymptotically central sequence of embeddings of D into itself, i.e., a sequence $\rho_k \colon D \to D$ of unital *-homomorphisms such that $\|\rho_k(x)y - y\rho_k(x)\| \to 0$ as $k \to \infty$ for all $x, y \in D$.

Identify D with $D_0^{\otimes\infty}$ where $D_0 \cong D$. Take a non-trivial projection p in D_0. For each natural number n let $E_n, E_n' \in D_0^{\otimes 2n}$ be as in Lemma 2.2 (corresponding to our non-trivial

projection p). Then $e_n \neq 0, E_n \neq 0$, and so $0 \neq f^{\otimes n} \neq 1$. Use (2.1) and Lemma 2.4 to find n such that $1 - E_n - E'_n \precsim p \otimes (1-p) \leq E_n$. It then follows from Lemma 2.1 that there is an injective unital *-homomorphism from $M_3 \otimes M_2$ into $D_0^{\otimes 2n} \subseteq D$. Composing this unital *-homomorphism with the unital *-homomorphism ρ_k yields an asymptotically central sequence of unital *-homomorphisms from $M_3 \otimes M_2$ into D. This shows that D is approximately divisible.

(ii) $\Rightarrow$ (iii). It is shown in [2] that a simple approximately divisible C^*-algebra is of real rank zero if and only if projections in the C^*-algebra separate the quasitraces. As quasitraces on a exact C^*-algebra are traces, [7], a result that applies to our case since strongly self-absorbing C^*-algebras are nuclear and hence exact, and since a strongly self-absorbing C^*-algebra has at most one tracial state, quasitraces are automatically separated by just one projection, say the unit.

(iii) $\Rightarrow$ (i). This is trivial. The only C^*-algebra of real rank zero that does not have a non-trivial projection is $\mathbb{C}$, the algebra of complex numbers. This C^*-algebra is not strongly self-absorbing by convention.

Finally, any simple approximately divisible C^*-algebra is $\mathcal{Z}$-absorbing, cf. [11]. □

Lemma 2.6 *Let D be a strongly self-absorbing C^*-algebra. Then $K_0(D)$ has a natural structure of commutative unital ring with unit $[1_D]$. If τ is a unital trace on D, then τ induces a morphism of unital rings $\tau_*\colon K_0(D) \to \mathbb{R}$.*

Proof: Fix an isomorphism $\gamma\colon D \otimes D \to D$. The multiplication on $K_0(D)$ is defined by composing $\gamma_*\colon K_0(D \otimes D) \to K_0(D)$ with the canonical map $K_0(D) \otimes K_0(D) \to K_0(D \otimes D)$. Since any two unital *-homomorphism from $D \otimes D$ to D are approximately unitarily equivalent, the above multiplication is well-defined and commutative. We leave the rest of proof for the reader, but note that if D has a unital trace, then $\tau \otimes \tau$ is the unique unital trace of $D \otimes D$. □

Proposition 2.7 *Let D be a strongly self-absorbing C^*-algebra. Suppose that D is quasidiagonal and that $K_0(D)$ is torsion free. Then either $K_0(D) \cong \mathbb{Z}$ or there is a* UHF *algebra B of infinite type such that $K_0(D) \cong K_0(B)$. If, in addition, D is assumed to contain a nontrivial projection, then $D \otimes B \cong D$, where B is as above.*

Proof: Since D is quasidiagonal it embeds unitally in the universal UHF algebra $B_{\mathbb{Q}}$ and $D \otimes B_{\mathbb{Q}} \cong B_{\mathbb{Q}}$, as explained in [5, Rem. 3.10]. The restriction of the unital trace of $B_{\mathbb{Q}}$ to D is denoted by τ. Thus we have an exact sequence

$$0 \longrightarrow H \longrightarrow K_0(D) \xrightarrow{\tau_*} \tau_* K_0(D) \longrightarrow 0$$

where H is the kernel of τ_*. Since $\mathbb{Z} \subseteq \tau_* K_0(D) \subseteq \mathbb{Q}$, and $K_0(D) \otimes \mathbb{Q} \cong \mathbb{Q}$, the map $\tau_* \otimes \mathrm{id}_{\mathbb{Q}}\colon K_0(D) \otimes \mathbb{Q} \to \tau_* K_0(D) \otimes \mathbb{Q}$ is an isomorphism. Therefore $H \otimes \mathbb{Q} = 0$ and so H is a torsion subgroup of $K_0(D)$. But we assumed that $K_0(D)$ is torsion free and hence $H = \{0\}$ and $\tau_*\colon K_0(D) \to \tau_* K_0(D) \subseteq \mathbb{Q}$ is an isomorphism of unital rings. The unital

subrings of $\mathbb{Q}$ are easily determined and well-known. They are parametrized by arbitary sets P of prime numbers. For each P the corresponding ring R_P consists of rational numbers r/s with r and s relatively prime and such that all prime factors of s are in P. If $P = \emptyset$ then $R_P = \mathbb{Z}$, otherwise R_P is isomorphic to the K_0-ring associated to a UHF algebra B of infinite type.

Suppose now that D contains a nontrivial projection. By Theorem 2.5, D has real rank zero and absorbs the Jiang-Su algebra $\mathcal{Z}$. In particular, $K_0(D)$ is not $\mathbb{Z}$ and is hence isomorphic (as a scaled abelian group) to $K_0(B)$ for some UHF-algebra B of infinite type. It follows from [8] that D has stable rank one and that $K_0(D)$ is weakly unperforated. Moreover, by [1, Sect. 6.9], $K_0(D)$ has the strict order induced by τ_*. The isomorphism $K_0(B) \cong K_0(D)$ of scaled abelian groups is therefore an order isomorphism, and by the properties of D established above we can conclude that B embeds unitally into D, whence $D \otimes B \cong D$. $\square$

Corollary 2.8 *Let D be a strongly self-absorbing C^*-algebra with torsion free K_0-group. Suppose that D contains a non-trivial projection and that D embeds unitally into the* UHF *algebra M_{p^∞} for some prime number p. Then $D \cong M_{p^\infty}$.*

Proof: By Proposition 2.7 there is a prime q such that M_{q^∞} in contained unitally in D and hence in M_{p^∞}. From this we deduce that $q = p$. Finally since $M_{p^\infty} \subseteq D \subseteq M_{p^\infty}$ we conclude that $D \cong M_{p^\infty}$. $\square$

3 Strongly self-absorbing algebras and K-theory

The class of strongly self-absorbing Kirchberg algebras satisfying the UCT was completely described in [11]. In this section we give properties and characterizations of strongly self-absorbing Kirchberg algebras which can be derived without assuming the UCT. For unital a C^*-algebra D we denote by ν_D the unital *-homomorphism $\mathbb{C} \to D$. When the C^*-algebra D is clear from context we will write ν instead of ν_D.

Proposition 3.1 *Let D be a strongly self-absorbing C^*-algebra. If D is not finite and the unital *-homomorphism $\mathbb{C} \to D$ induces a surjection $K^0(D) \to K^0(\mathbb{C})$, then $D \cong O_\infty$.*

Proof: By [11, Prop. 5.12], two strongly self-absorbing C^*-algebras are isomorphic if and only if they embed unitally into each other. Thus it suffices to show the existence of unital *-homomorphisms $\mathcal{O}_\infty \to D$ and $D \to \mathcal{O}_\infty$ Since D is not finite, it must be a Kirchberg algebra, see [11, Sec. 1], and hence $\mathcal{O}_\infty$ embeds unitally in D by [10, Prop. 4.2.3]. It remains to show that D embeds unitally in $\mathcal{O}_\infty$.

By assumption, the map $\nu^*\colon KK(D, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C})$ is surjective. By multiplying with the KK-equivalence class given by the unital morphism $\mathbb{C} \to \mathcal{O}_\infty$, we obtain that the map $\nu^*\colon KK(D, \mathcal{O}_\infty) \to KK(\mathbb{C}, \mathcal{O}_\infty)$ is surjective. If $\varphi\colon D \to \mathcal{O}_\infty \otimes \mathcal{K}$ is a *-homomorphism, then, after identifying $KK(\mathbb{C}, \mathcal{O}_\infty) \cong K_0(\mathcal{O}_\infty)$, the map ν^* sends $[\varphi]$ to the class $[\varphi(1_D)] \in K_0(\mathcal{O}_\infty)$. By [10, Thm. 8.3.3] each element of $KK(D, \mathcal{O}_\infty)$ is represented

by a *-homomorphism. Therefore, by the surjectivity of ν^*, there is a *-homomorphism $\varphi\colon D \to \mathcal{O}_\infty \otimes \mathcal{K}$ such that $[\varphi(1_D)] = [1_{\mathcal{O}_\infty}]$. Since these are both full projections, by [10, Prop. 4.1.4] there is a partial isometry $v \in \mathcal{O}_\infty \otimes \mathcal{K}$ such that $v^*v = \varphi(1_D)$ and $vv^* = 1_{\mathcal{O}_\infty}$. Then $v\varphi v^*$ is a unital embedding $D \to \mathcal{O}_\infty$. □

Remark 3.2 Note that the isomorphism $D \cong \mathcal{O}_\infty$ was obtained without assuming that D satisfies the UCT. Let us argue that assumptions of Proposition 3.1 are natural. Let A and B be unital C^*-algebras and let $\nu\colon \mathbb{C} \to A$ and $\nu\colon \mathbb{C} \to B$ be the corresponding unital *-homomorphisms. The condition that there is a morphism of pointed groups $(K_0(A), [1_A]) \to (K_0(B), [1_B])$ can be viewed as the condition that the diagram

$$\begin{tikzcd} A & & B \\ & \mathbb{C} \arrow[ul, "\nu"] \arrow[ur, "\nu"'] & \end{tikzcd}$$

can be completed to a commutative diagram after passing to K-theory:

$$\begin{tikzcd} K_0(A) \arrow[rr, dashed] & & K_0(B) \\ & K_0(\mathbb{C}) \arrow[ul, "\nu_*"] \arrow[ur, "\nu_*"'] & \end{tikzcd}$$

It would then be completely natural to use K-homology instead of K-theory and ask that the first diagram can be completed to a commutative diagram after passing to K-homology.

$$\begin{tikzcd} K^0(A) \arrow[dr, "\nu^*"'] & & K^0(B) \arrow[ll, dashed] \arrow[dl, "\nu^*"] \\ & K^0(\mathbb{C}) & \end{tikzcd}$$

Now let us observe that the condition, imposed in Proposition 3.1, that $\nu^*\colon K^0(D) \to K^0(\mathbb{C})$ is surjective clearly is equivalent to the existence of a commutative diagram

$$\begin{tikzcd} K^0(\mathcal{O}_\infty) \arrow[dr, "\nu^*"'] & & K^0(D) \arrow[ll, "\alpha"'] \arrow[dl, "\nu^*"] \\ & K^0(\mathbb{C}) & \end{tikzcd}$$

where α is a surjective morphism.

If D satisfies the UCT, then the condition above can be translated in terms of K-theory as follows. Since the commutative diagram

$$\begin{tikzcd} K^0(D) \arrow[r] \arrow[d, "\nu^*"'] & \mathrm{Hom}(K_0(D), \mathbb{Z}) \arrow[d] \\ K^0(\mathbb{C}) \arrow[r] & \mathrm{Hom}(K_0(\mathbb{C}), \mathbb{Z}) \end{tikzcd}$$

has surjective horizontal arrows, the assumption on K-homology in Proposition 3.1 is equivalent the existence a group homomorphism $K_0(D) \to \mathbb{Z}$ which maps $[1_D]$ to 1. This is obviously equivalent to the condition that $[1_D]$ is an infinite order element of $K_0(D)$ and that the subgroup that it generates, $\mathbb{Z}[1_D]$, is a direct summand of $K_0(D)$.

Our next goal is to show that for a unital Kirchberg algebra the property of being strongly self-absorbing is purely a KK-theoretical condition. Let

$$C_\nu = \{f\colon [0,1] \to D \mid f(0) \in \mathbb{C}1_D, \quad f(1) = 0\}$$

be the mapping cone of the unital *-homomorphisms $\nu\colon \mathbb{C} \to D$.

Proposition 3.3 *Let D be a unital Kirchberg algebra. Then D is strongly self-absorbing if and only if $C_\nu \otimes D$ is KK-equivalent to zero.*

Proof: We begin with a general observation. For a *-homomorphism $\varphi\colon A \to B$ of separable C^*-algebras and any separable C^*-algebra C, there is an exact Puppe sequence in KK-theory ([1, Thm. 19.4.3]):

$$\begin{tikzcd} KK(B,C) \arrow[r, "\varphi^*"] & KK(A,C) \arrow[r] & KK(C_\varphi, C) \arrow[d] \\ KK_1(C_\varphi, C) \arrow[u] & KK_1(A,C) \arrow[l] & KK_1(B,C) \arrow[l, "\varphi^*"'] \end{tikzcd}$$

It is apparent that $[\varphi] \in KK(A,B)^{-1}$ if and only if composition with $[\varphi] \in KK(A,B)$ induces a bijection $\varphi^*\colon KK(B,C) \to KK(A,C)$ for any separable C^*-algebra C, or equivalently, for just $C = A$ and $C = B$. Therefore, by the exactness of the Puppe sequence, we see that that φ induces a KK-equivalence if and only if its mapping cone C^*-algebra C_φ is KK-contractible.

By applying this observation to the unital *-homomorphism $\nu \otimes \mathrm{id}_D\colon D \to D \otimes D$ we deduce that $\nu \otimes \mathrm{id}_D$ induces a KK-equivalence if and only if its mapping cone $C_{\nu\otimes\mathrm{id}_D} \cong C_\nu \otimes D$ is KK-contractible. Suppose now that D is a strongly self-absorbing Kirchberg algebra. Then $\nu \otimes \mathrm{id}_D$ is asymptotically unitarily equivalent to a an isomorphism by [5, Thm. 2.2] and hence $\nu \otimes \mathrm{id}_D$ induces a KK-equivalence. Conversely, if $\nu \otimes \mathrm{id}_D$ induces a KK-equivalence, then $\nu \otimes \mathrm{id}_D$ is asymptotically unitarily equivalent to an isomorphism $D \to D \otimes D$ by [10, Thm. 8.3.3] and hence D is strongly self-absorbing. □

We have the following result related to Proposition 3.3.

Proposition 3.4 *Let D be a unital Kirchberg algebra such that $D \cong D \otimes D$. The following assertions are equivalent:*

(i) *D is strongly self-absorbing.*

(ii) $KK(C_\nu, SD) = 0$.

(iii) $KK(C_\nu, D \otimes A) = 0$ *for all separable* C^**-algebras* A.

(iv) *The map* $KK(D, D \otimes A) \to KK(\mathbb{C}, D \otimes A)$ *is bijective for all separable* C^**-algebras* A.

Proof: (iii) $\Leftrightarrow$ (iv). This equivalence is verified by using the Puppe sequence associated to $\nu\colon \mathbb{C} \to D$, arguing as in the proof of Proposition 3.3.

(i) $\Rightarrow$ (iv). This implication is proved in [5, Thm. 3.4].

(iii) $\Rightarrow$ (ii). This follows by taking $A = S\mathbb{C}$ in (iii).

(ii) $\Rightarrow$ (i). Fix an isomorphism $\gamma\colon D \to D \otimes D$. Since $KK_1(C_\nu, D \otimes D) = 0$ by hypothesis, it follows from the Puppe sequence that the map $\nu^*\colon KK(D, D \otimes D) \to KK(\mathbb{C}, D \otimes D)$ is injective. Therefore γ and $\nu \otimes \mathrm{id}_D$ induce the same class in $KK(D, D \otimes D)$ since they are both unital. It follows that $\nu \otimes \mathrm{id}_D$ is asymptotically unitarily equivalent to γ and so D is strongly self-absorbing. $\square$

Corollary 3.5 *Let D be a unital Kirchberg algebra such that $D \cong D \otimes D$. Then D is strongly self-absorbing if and only if* $\pi_2 \operatorname{Aut}(D) = 0$.

Proof: Since $\pi_2 \operatorname{Aut}(D) \cong KK(C_\nu, SD)$ by [5, Cor. 3.1], the conclusion follows from Proposition 3.4. $\square$

It was shown in [4, Prop. 4.1] that if a unital Kirchberg algebra satisfies the UCT, then D is strongly self-absorbing if and only if the homotopy classes $[X, \operatorname{Aut}(D)]$ reduces to singleton for any path connected compact metrizable space X.

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Department of Mathematics, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark

E-mail address: rordam@math.ku.dk
Internet home page: www.math.ku.dk/~rordam

Department of Mathematics, Purdue University West Lafayette, IN 47906, USA

E-mail address: mdd@math.purdue.edu
Internet home page: www.math.purdue.edu/~mdd